Math 451: Introduction to General Topology
Lecture 7

Subsequences. A subsequence (x_{n_k}) of a sequence (x_n) is another sequence (y_k), indexed by k, such that the k^{th} member is the n_k^{th} member of (x_n), i.e. $y_k = x_{n_k}$, where (n_k) is a strictly increasing sequence of natural numbers. In particular, $n_k \geq k$.

Prop. For a sequence (x_n) in a metric space (X,d) and $x \in X$, TFAE:

(1) (x_n) converges to x.

(2) Every subsequence (x_{n_k}) converges to x.

(3) Every subsequence $(x_{n_k})_{k\in\mathbb{N}}$ has a further subsequence $(x_{n_{k_\ell}})_{\ell\in\mathbb{N}}$ converges to x.

Proof. (1)$\Rightarrow$(2): Trivial.

(2)$\Rightarrow$(1): We prove the contrapositive, i.e. not (1) $\Rightarrow$ not (2). Suppose (x_n) doesn't converge to x, i.e. $\exists$ open $U \ni x$ such that not $\forall^\infty_n\ x_n \in U$, i.e. $\exists^\infty_n\ x_n \notin U$. Let (n_k) be the sequence of those infinitely many n's, i.e. $x_{n_k} \notin U$ for all $k \in \mathbb{N}$. Thus, (x_{n_k}) doesn't converge to x.

(1)$\Leftrightarrow$(3). HW. $\boxed{\text{QED}}$

Closure and characterization of closed sets via limits.

Def. Let X be a metric space and $Y \subseteq X$. We call the smallest (with respect to inclusion $\subseteq$) closed set containing Y the closure of Y, denoted $\overline{Y}$. Such a set exists, namely:

$$\overline{Y} := \bigcap \{ C \subseteq X : C \text{ closed},\ C \supseteq Y \},$$

noting that the right hand side is closed being an intersection of closed sets. Since X is closed and contains Y, $\{C \subseteq X : C \text{ closed}, C \supseteq Y\} \neq \emptyset$, and $Y \subseteq \overline{Y}$ by definition.

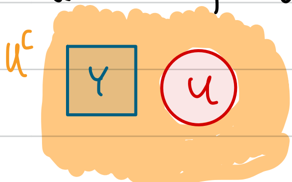
Def. Let X be a metric space and $Y \subseteq X$. We say that a point $x \in X$ adheres to Y (or is a closure point of Y) if every open neighbourhood $U \ni x$ meets Y (i.e. $U \cap Y \neq \emptyset$).

Examples. (a) In $\mathbb{R}$, 1 adheres to $(0,1)$. Also, $\frac{1}{2}$ adheres to $(0,1)$.
(b) In $\mathbb{R}$, every real $r \in \mathbb{R}$ adheres to $\mathbb{Q}$.

Prop. Let X be a metric space and $Y \subseteq X$. Then
$$\overline{Y} = \{x \in X : x \text{ adheres to } Y\}.$$

Proof. $\supseteq$: Let $x \in X$ adhere to Y. If $x \in \overline{Y}^c$ then because $\overline{Y}^c$ is open, there is an open $U \ni x$ with $U \subseteq \overline{Y}^c$, i.e. $U \cap \overline{Y} = \emptyset$, hence $U \cap Y = \emptyset$, contradicting that x adheres to Y.

$\subseteq$: Let $x \in \overline{Y}$, and take any open $U \ni x$. We need to show $U \cap Y \neq \emptyset$. But if $U \cap Y = \emptyset$,



i.e. $U^c \supseteq Y$ and U^c is closed, so $\overline{Y} \subseteq U^c$ being the smallest such set. But this means $U \cap \overline{Y} = \emptyset$, contradicting $x \in U \cap \overline{Y}$. $\boxed{\text{QED}}$

Prop (characterization of adherence via limits). Let (X,d) be a metric space, $Y \subseteq X$, $x \in X$. TFAE:
(1) $x \in \overline{Y}$.
(2) x adheres to Y.
(3) There is a sequence $(y_n) \subseteq Y$ with $\lim_{n\to\infty} y_n = x$.

Proof. (3)$\Rightarrow$(2). If $(y_n) \subseteq Y$ and $x = \lim_{n\to\infty} y_n$ then every open $U \ni x$ meets $\{y_n : n \in \mathbb{N}\} \subseteq Y$, so it also meets Y, hence x adheres to Y.

(2)$\Rightarrow$(3). Suppose x adheres to Y. Then for each $n \in \mathbb{N}^+$, there is $y_n \in B_{\frac{1}{n}}(x) \cap Y$ (AC is used here to define a sequence of such choices), so we obtain a sequence $(y_n) \subseteq Y$ such that $d(y_n, x) < \frac{1}{n} \to 0$ as $n \to \infty$, so $\lim_{n\to\infty} y_n = x$. $\boxed{\text{QED}}$

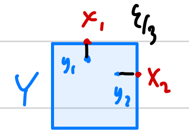
Characterization of closed sets. Let X be a metric space and $Y \subseteq X$. TFAE:
(1) Y is closed (i.e. Y^c is open).
(2) $Y = \overline{Y}$.
(3) Y contains all its adherent points.
(4) For any sequence $(y_n) \subseteq Y$ if $\lim_{n\to\infty} y_n$ exists then it is in Y.

Proof. HW

Cor. In a metric space X, for $Y \subseteq X$, $\operatorname{diam}(Y) = \operatorname{diam}(\overline{Y})$.

Proof. $\leq$ is trivial, so we show $\geq$. We give ourselves an $\varepsilon > 0$ room and show that $\operatorname{diam}(Y) \geq \operatorname{diam}(\overline{Y}) - \varepsilon$ for arbitrary $\varepsilon > 0$. By def, $\exists\, x_1, x_2 \in \overline{Y}$ with $d(x_1, x_2) \geq \sup_{z_1, z_2 \in \overline{Y}} d(z_1, z_2) - \varepsilon/3$. But $x_1, x_2 \in \overline{Y}$, so $B_{\varepsilon/3}(x_1) \cap Y \neq \emptyset$ and $B_{\varepsilon/3}(x_2) \cap Y \neq \emptyset$, so take $y_i \in B_{\varepsilon/3}(x_i)$, for $i = 1, 2$. Then

$$\operatorname{diam}(\overline{Y}) - \varepsilon/3 \leq d(x_1, x_2) \leq d(x_1, y_1) + d(y_1, y_2) + d(y_2, x_2) \leq d(y_1, y_2) + \frac{2}{3}\varepsilon.$$



Hence $d(y_1, y_2) \geq \operatorname{diam}(\overline{Y}) - \varepsilon$, hence $\operatorname{diam}(Y) \geq \operatorname{diam}(\overline{Y}) - \varepsilon$. $\boxed{\text{QED}}$

Cauchy sequences and completeness.

Def. A sequence (x_n) in a metric space (X, d) is called Cauchy if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ $\forall n, m \geq N$, $d(x_n, x_m) \leq \varepsilon$, i.e. the tails of the sequence are tighter and tighter together. This is equivalent to saying that $\operatorname{diam}(\{x_n, x_{n+1}, x_{n+2}, \dots\}) \to 0$ as $n \to \infty$.

Obs. Cauchy sequences are bdd.

Examples. (a) Convergent sequences are Cauchy.

Proof. If (x_n) converges to x, then $d(x_n, x) \to 0$, so $\forall \varepsilon > 0$ a tail of the sequence is in $B_\varepsilon(x)$, hence has diameter $\leq 2\varepsilon$. $\square$

(b) A sequence (x_n) is called contractive if there is $\alpha \in (0, 1)$ such that for all $n \in \mathbb{N}$:

$$d(x_{n+1}, x_{n+2}) \leq \alpha\, d(x_n, x_{n+1}).$$

Prop. Contractive sequences are Cauchy.

Proof. HW

Caution. The weaker condition $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$ for all n, does not imply that (x_n) is Cauchy. Take, e.g., $x_n := \sum_{k=1}^{n} \frac{1}{k}$, then

$$d(x_{n+1}, x_{n+2}) = \left| \sum_{k=1}^{n+2} \frac{1}{k} - \sum_{k=1}^{n+1} \frac{1}{k} \right| = \frac{1}{n+2} < \frac{1}{n+1} = d(x_n, x_{n+1}).$$

But (x_n) doesn't converge as $\sum_{k \in \mathbb{N}^+} \frac{1}{k}$ doesn't converge.

Prop. Let X be a metric space and $(x_n) \subseteq X$ be a Cauchy sequence. If some subsequence (x_{n_k}) converges to a point $x \in X$ then the whole (x_n) converges to x.

Proof. Suppose $\lim_{k\to\infty} x_{n_k} = x$. Fix $\varepsilon > 0$. Then (x_n) being Cauchy gives $n \in \mathbb{N}$, such that $\operatorname{diam}(\{x_n, x_{n+1}, x_{n+2}, \dots\}) \leq \varepsilon/3$. Then because $\lim_{k\to\infty} x_{n_k} = x$, there is some $n_k \geq n$ with $d(x_{n_k}, x) < \varepsilon/3$. But then $\forall\, m \geq n$, we have

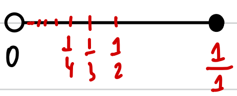
$$d(x_m, x) \leq d(x_m, x_n) + d(x_n, x_{n_k}) + d(x_{n_k}, x) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Thus, $\lim_{m\to\infty} x_m = x$. $\boxed{\text{QED}}$

Def. A metric space X is called *complete* if every Cauchy sequence in it converges to some point in X.

Examples. (a) Let X be a 0-1 metric space, i.e. d is the 0-1 metric on a set X. Then the only Cauchy sequences in X are the *eventually constant sequences*, i.e. sequences (x_n) such that $\exists x \in X$ with $\forall^\infty_n\ x_n = x$. Hence every Cauchy sequence here converges to a point in X, thus X is complete.

(b) The space $X = (0,1]$ with the usual metric d is incomplete because the sequence $(\frac{1}{n}) \subseteq (0,1]$ is Cauchy (indeed $\operatorname{diam}(\{\frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}, \dots\}) = \frac{1}{n} \to 0$ as $n \to \infty$.



But $(\frac{1}{n})$ does not converge in $X := (0,1]$.

(c) The space $X := \mathbb{Q}$ is incomplete because $\exists\, (x_n) \subseteq \mathbb{Q}$ converging in $\mathbb{R}$ to $\sqrt{2}$, so it is Cauchy in $\mathbb{R}$, hence in $\mathbb{Q}$, but doesn't converge in $\mathbb{Q}$ since $\sqrt{2} \notin \mathbb{Q}$ and limits are unique in $\mathbb{R}$.